Extremal And Probabilistic Graph Theory
Note 12 April 7th Thursday

- We will consider trees,paths and cycles in coming lectures.

Definition 1. For a path P, $|P|$ is the length of P, *i.e.* $\sharp$ edges in P; Let P_t denote the path of length t.

Conjecture. (Erdös-Sós) Let T be a tree on $t+1$ vertices, then $ex(n,T) = \frac{(t-1)n}{2}$.
Lower bound: Consider the graph G consisting of vertex-disjoint $K_t^{\prime}s$, this shows $ex(n,T) \geq \frac{(t-1)n}{2}$. (This conjecture was confirmed by Ajtai-Komlós-Simonovits-Szemerédi.)
In below, we will prove a much weaker (halfway) bound.

Lemma 1. If G has n vertices and m edges, then G has a subgraph H with $\delta(H) > \frac{m}{n}$.

Proof: If $\delta(G) > \frac{m}{n}$,then we are done.
Otherwise we pick a vertex v with $d_G(v) \leq \frac{m}{n}$. Let $G^{\prime}=G-\{v\}$, then the new graph $G^{\prime}$ has average-degree$\geq \frac{2(m-\frac{m}{n})}{n-1} = \frac{2m}{n}$, which is non-decreasing.
Continuing in this way, we end up with a required subgraph. ∎

Theorem 1. For any tree T on $t+1$ vertices, $ex(n,T) < (t-1)n$.

Proof: Suppose G is T-free graph with $\geq (t-1)n$ edges. By Lemma 1, G has a subgraph H with $\delta(H) \geq t$.
(Exersice: Any graph H with min-degree $\geq t$ contains a copy of any tree on $t+1$ vertices.)
(Hint: Find a good linear ordering of $V(T)$ and greedy embed T into H.) ∎

Fact 1. Any graph G has a path of length $\geq \delta(G)$.
(Bound is tight: Graphs consisting of disjoint $K_t^{\prime}s$.)

Theorem 2. If G is connected, then G has a path of size $\geq min\{n, 2\delta(G)+1\}$.

Definition 2. Let P be a path in G from u to v, for $x \in V(P)$, denote x^- and x^+ to be the immediate predecessor and immediate successor on P. For $S \subseteq V(P), S^+ = \{x^+ : x \in S\}$. Similarly, define this for S^-.

Fact 2. Let P be a largest path in G from u to v. For $w \in N(v) \cap V(P)$, the path $P^{\prime} = P - \{ww^+\} + \{wv\}$ is also a largest path in G.

Definition 3. This transformation from P to $P^{\prime}$ is call a Pósa rotation.

Definition 4. A path(or a cycle) is call *Hamiltonian*, if it contains all vertices of the graph. A graph G is Hamiltonian, if G has a such cycle.

Fact 3. Let G be connected and P ba a largest path in G. If there exists a cycle C with $V(C) = V(P)$, then G is Hamiltonian.

Proof: If $V(C) \neq V(G)$, as G is connected, there exists some $a \in V(C)$ and $b \in V(G) \setminus V(C)$ s.t. $ab \in E(G)$. Then we can find a longer path than P, a contradiction. ∎

Proof of Theorem 2: Let P be a longest path in G, say $P = x_0x_1...x_m$, since $|P|$ is maximum, $N(x_0), N(x_m) \subseteq V(P)$, assume for a contradiction that $|V(P)| < n$ and $V(P) \leq 2\delta(G)$. ∎

Claim: $\exists i \in \{0, 1...m-1\}$ s.t. $x_0x_{i+1}, x_ix_m \in E(G)$.
Proof: Suppose not, then $N(x_0) \cap N(x_m)^+ = \emptyset$, also $x_0 \notin N(x_0) \cup N(x_m)^+$
$\implies |V(P)| \geq 1 + |N(x_0)| + |N(x_m)|^+ = 1 + d(x_0) + d(x_m) \geq 1 + 2\delta(G)$.
But $|V(P)| \leq 2|\delta(G)|$, a contradiction.This proves the claim.
Then we can find a cycle $C, C = P - \{x_ix_{i+1}\} + \{x_0x_{i+1}\} + \{x_ix_m\}$ s.t. $V(C) = V(P)$. ∎

Theorem 3.(Erdös-Gallai) For $n \geq t$, $ex(n, P_t) \leq \dfrac{(t-1)n}{2}$.
Moreover, the extremal graph is unique, that is a disjoint union of $K_t\ 's$.

Remark: This is a special case of the Erdös-Sós Conjecture.

Proof: By induction on n. It is trivial for $n = t$.

For $n > t$, let G be a P_t-free graph on n vertices, we want to show $e(G) \leq \dfrac{(t-1)n}{2}$.
We may assume that G is connected. (Otherwise,we apply induction on each connected component of G).
If $\delta(G) \geq \dfrac{t}{2}$, by Theorem 2, we know that G has a path of size $\geq min\{n, t+1\} \geq t+1$, a contradiction to G is P_t-free.
So there is a vertex v of degree$\leq \dfrac{t-1}{2}$. Let $G' = G - \{v\}$, by induction, $e(G') \leq \dfrac{(t-1)(n-1)}{2}$.
Thus, $e(G) = e(G') + d_v \leq \dfrac{(t-1)n}{2}$.
The "moreover" part is left as an exercise. ∎

Theorem 4. (Ore's Theorem) Let G be an n-vertex graph, such that any non-adjacent vertices u, v satisfy $d_G(u) + d_G(v) \geq n$. Then G is Hamiltonian.

Proof: First, such G must be connected, let P be a largest path from u to v, $uv \notin E(G)$, $d_G(u) + d_G(v) \geq n \implies N(u) \cap N(v)^+ \neq \emptyset$.(By Fact 3) ∎

Corollary 2.(Dirac's Theorem) If $\delta(G) \geq \dfrac{n}{2}$, then G is Hamiltonian.
The above proof tells us more:

Fact 4. Let $uv \notin E(G)$ with $d_G(u) + d_G(v) \geq n$, then G is Hamiltonian iff $G + \{uv\}$ is Hamiltonian.

Definition 5. The *closure* of a graph G is the graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least n until no such pair exists.

Lemma 2. The closure of G is well-defined.
(The ordering of adding edges will not affect the final graph).

Theorem 5.(Bondy-Chvátal) A graph is Hamiltonian iff its closure is Hamiltonian.

Remark: Theorem 4 can be derived from Theorem 5.